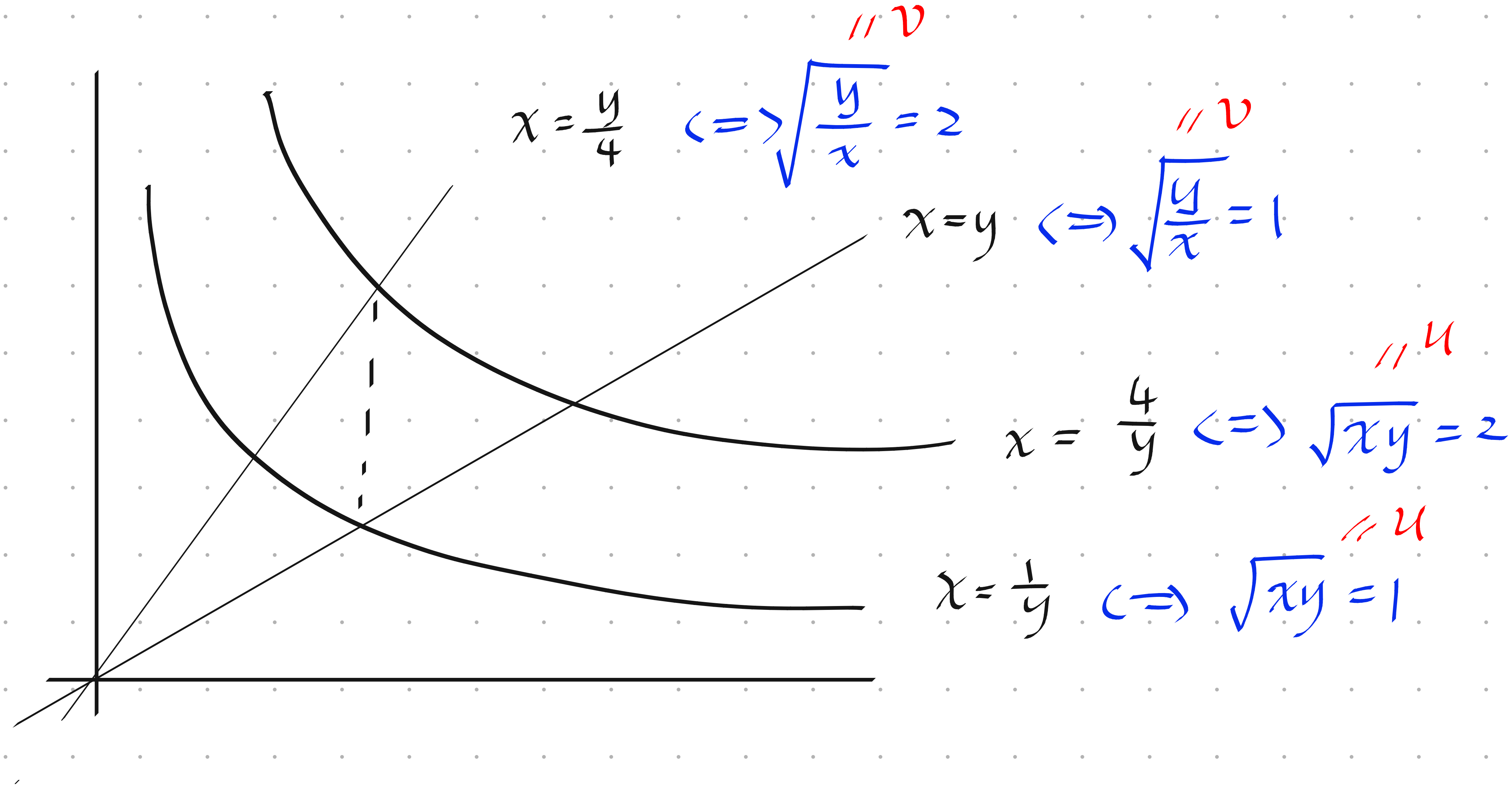
Assignment 6 Solution.

Q15

Given $x = \frac{u}{v}$, $y = uv$ $\Rightarrow$ $u = \sqrt{xy}$ and $v = \sqrt{\frac{y}{x}}$.



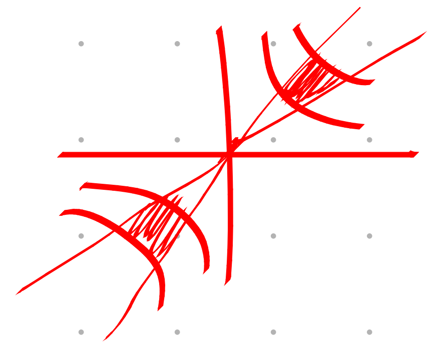
$$\Rightarrow \int_1^2 \int_{1/y}^{y} (x^2+y^2)\,dx\,dy + \int_1^2 \int_{y/4}^{4/y} (x^2+y^2)\,dx\,dy$$

$$= \int_1^2 \int_1^2 \left(\frac{u^2}{v^2} + u^2v^2\right) \cdot \underline{\det J_\Phi}\; du\,dv \qquad J_\Phi = \begin{pmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{pmatrix}$$

$$= \int_1^2 \int_1^2 \left(\frac{u^2}{v^2} + u^2v^2\right) \cdot \frac{2u}{v}\; du\,dv$$

$\vdots$

$= \frac{225}{16}$ Answer $= \frac{225}{16} \times 2 = \frac{225}{8}$ $\because$



Q20

$D := \{(x,y,z) : 1 \le x \le 2,\ 0 \le xy \le 2,\ 0 \le z \le 1\}$.

$u = x,\ v = xy,\ w = 3z \iff x = u,\ y = \frac{v}{u},\ z = \frac{1}{3}w$

$\Rightarrow D := \{(u,v,w) : 1 \le u \le 2,\ 0 \le v \le 2,\ 0 \le w \le 3\}$.

$$\det J_\Phi = \det \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{pmatrix} 1 & 0 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

$$= \frac{1}{3u}$$

$$\Rightarrow \iiint_D (x^2y + 3xyz)\, dx\, dy\, dz$$

$$= \int_1^2 \int_0^2 \int_0^3 \left(u^2 \cdot \frac{v}{u} + 3 \cdot v \cdot \frac{1}{3}w\right) \frac{1}{3u}\, dw\, dv\, du$$

$\vdots$

$= 2 + 3\ln 2$

Q12 (a)

$x = r\cos\theta, \quad y = r\sin\theta$

$$y\cot\beta \le x \le \sqrt{a^2-y^2} \qquad a>0, \quad 0<\beta<\frac{\pi}{2}.$$

$$0 \le y \le a\sin\beta$$

$$\Rightarrow \quad r\sin\theta\cot\beta \le r\cos\theta \le \sqrt{a^2-r^2\sin\theta}$$

$$0 \le r\sin\theta \le a\sin\beta.$$

$$\tan\theta \le \tan\beta$$

$$\Rightarrow \quad \theta \le \beta.$$

$$r^2\cos^2\theta \le a^2 - r^2\sin^2\theta$$

$$r^2 \le a^2$$

$$r \le a$$

$$\Downarrow$$

$$0 \le r\sin\beta \le a\sin\beta \quad \Rightarrow \quad 0 \le r \le a.$$

Since $\tan\theta = \frac{y}{x} \Rightarrow$ when $y=0$, $\tan\theta = 0 \Rightarrow \theta = 0$

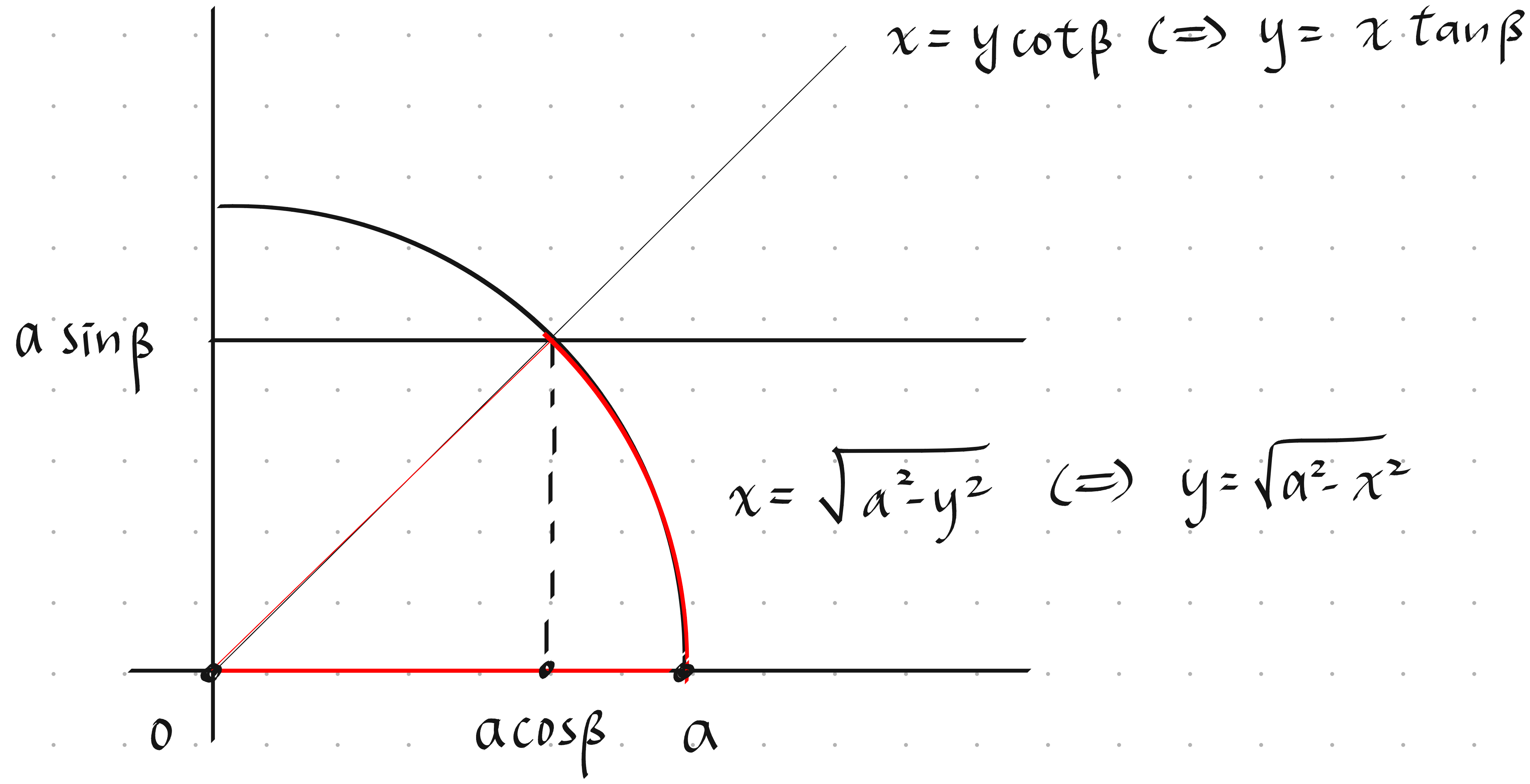
$$\Rightarrow \quad 0 \le \theta \le \beta$$

$$\Rightarrow \int_0^{a\sin\beta}\int_{y\cot\beta}^{\sqrt{a^2-y^2}} \ln(x^2+y^2)\,dx\,dy = \int_0^{\beta}\int_0^{a} r\ln(r^2)\,dr\,d\theta$$

$$= \vdots$$

$$= a^2\beta\left(\ln a - \frac{1}{2}\right)$$

Q12 (b)



$$\int_0^{a\cos\beta}\int_0^{x\tan\beta} \ln(x^2+y^2)\,dy\,dx$$

$$+\int_{a\cos\beta}^{a}\int_0^{\sqrt{a^2-x^2}} \ln(x^2+y^2)\,dy\,dx$$

Q14

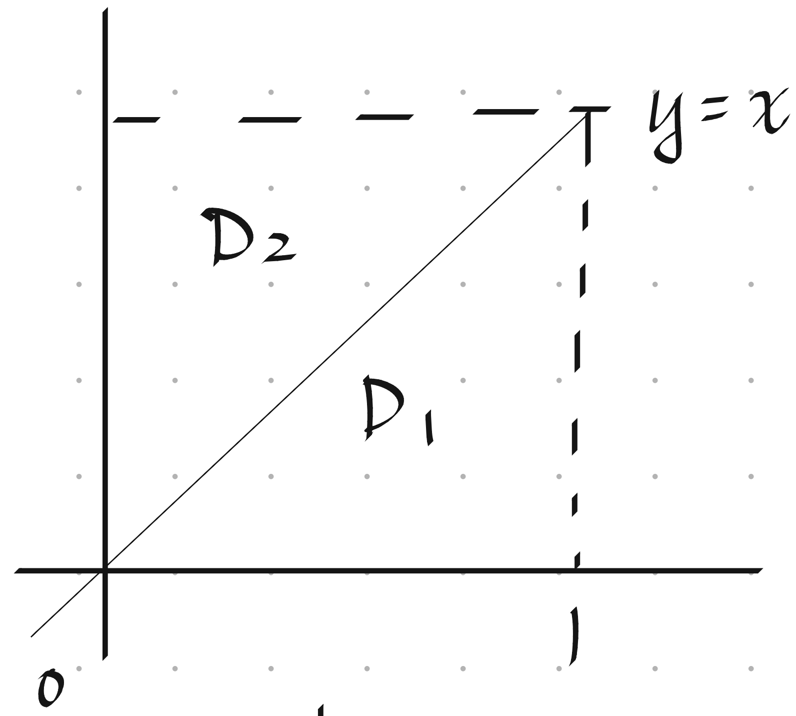
$$\int_0^1 f(x)\int_0^x g(x-y)\, f(y)\, dy\, dx$$

$$= \int_0^1\int_0^x f(x)\, g(x-y)\, f(y)\, dy\, dx$$

$$\parallel$$

$$\int_0^1\int_y^1 f(x)\, g(x-y)\, f(y)\, dx\, dy$$

$$= \int_0^1 f(y)\int_y^1 g(x-y)\, f(x)\, dx\, dy$$



$$\downarrow$$

$$\underbrace{\iint_{D_2} = \frac{1}{2}\iint_{D_1+D_2}}$$

$$\int_0^1\int_0^x g(x-y)\, f(y)\, dy\, dx = \frac{1}{2}\int_0^1\int_0^1 g(|x-y|)\, f(x) f(y)\, dx\, dy \quad //$$

$\because$ $x-y>0 \Rightarrow y<x \Rightarrow D_1$

$x-y<0 \Rightarrow y>x \Rightarrow D_2$